THE SOLUTION OF THE BOUSSINESQ PROBLEM FOR A HALF-SPACE WHOSE MODULUS OF ELASTICITY IS A POWER FUNCTION OF THE DEPTH

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The exact solution of the problem given in the title is constructed by the methods of integral transforms and analytic functions. This problem has arisen in connection with the theory of the linearly-deformable foundations, characterized by a power kernel. The papers [1 - 6] are devoted to integral equations of the first kind. In them, one assumes that such a kernel corresponds to the elastic half-space whose modulus of elasticity is a power function of the depth. For the determination of the exact form of the kernel one has to solve Flamant's problem (in the plane elasticity theory) and Boussinesq's problem (in the three-dimensional theory). The solution of Flamant's problem is given in [7, 2] and a partial solution of the Boussinesq problem is given in [11] (only the displacements on the boundary of the half-space are calculated). An attempt for the computation of the elastic field in the half-space can be found in a paper where the initial terms of the series, representing the solution in spherical coordinates, are computed but there is no general formula for them and the convergence is not established. (*)

In this paper we will use cylindrical coordinates; the solution is expressed in closed form in terms of known higher transcendental functions. The formulas of the solution are obtained applying A. Ia. Aleksandrov's [8] process of transforming plane problems of the theory of elasticity into axially symmetric ones and conversely, and also by applying integral transforms whose kernels are Whittaker functions.

1. We consider an elastic nonhomogeneous half-space, bounded by the plane $z = 0$, whose modulus of elasticity is a power function of the depth, i. e. $\mu = Kz^k$, where K is a constant, while Poisson's ratio ν is constant. A concentrated force P acts along the z-axis on the boundary $z = 0$ of the half-space. In order to solve the formulated problem we apply the following formulas [8]:

$$p(x) = \frac{1}{\pi}\frac{\partial}{\partial x}\int_0^x \frac{p^*(r)\,r\,dr}{\sqrt{x^2 - r^2}} \tag{1.1}$$

$$\sigma_r^* + \sigma_\theta^* = \int_{-r}^{r} (\sigma_x + \sigma_y)\frac{dx}{\sqrt{r^2 - x^2}}, \qquad \sigma_z^* = \int_{-r}^{r} \sigma_z \frac{dx}{\sqrt{r^2 - x^2}} \tag{1.2}$$

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*) Belik, G. I., Some three-dimensional problems of the calculation of constructions on generalized elastic foundations. Author's essay of candidate dissertation. Dnepropetrovsk, 1963.

$$\sigma_r^* - \sigma_\theta^* = \int_{-r}^{r} (\sigma_x - \sigma_y) \frac{2x^2 - r^2}{r^2} \frac{dx}{\sqrt{r^2 - x^2}}, \quad u^* = \int_{-r}^{r} u_x \frac{x}{r} \frac{dx}{\sqrt{r^2 - x^2}}$$

$$\tau_{rz}^* = \int_{-r}^{r} \tau_{xr} \frac{x}{r} \frac{dx}{\sqrt{r^2 - x^2}}, \quad w^* = \int_{-r}^{r} u_z \frac{dx}{\sqrt{r^2 - x^2}}$$

$$\sigma_y = \nu (\sigma_x + \sigma_z)$$

Here and in the following, $p(x)$, σ_x, σ_y, σ_z, τ_{xz}, u_x, u_y are, respectively, the load the stresses and the displacements of the plane state while the asterisk denotes the quantities of the axially symmetric state. For a given load of the axially symmetric state (the concentrated normal force P) we determine by formula (1.1) the load corresponding to the plane problem. One finds easily that

$$p(x) = -(2\pi^2)^{-1} P x^{-2} \tag{1.3}$$

As for the stress and displacement components of the plane state, corresponding to the distributed load $p(x)$, they can be found with the aid of the solution of Flamant's problem for a nonhomogeneous plate, whose modulus of elasticity is $\mu = Kz^k$, where $0 \leqslant k < 1$.

The solution of Flamant's problem in polar coordinates is taken from [11]

$$\sigma_x = -CPz^k x^2 \varphi, \quad \sigma_z = -CPz^{k+2}\varphi, \quad \tau_{xz} = -CPz^{k+1} x\varphi$$

$$u_x = \frac{PA}{\psi}\left[x \cos\left(q \operatorname{arc\,tg} \frac{x}{z}\right) - \frac{qz}{1+k} \sin\left(q \operatorname{arc\,tg} \frac{x}{z}\right)\right] \tag{1.4}$$

$$u_z = \frac{PA}{\psi}\left[z \cos\left(q \operatorname{arc\,tg} \frac{x}{z}\right) + \frac{qx}{1+k} \sin\left(q \operatorname{arc\,tg} \frac{x}{z}\right)\right]$$

$$\varphi = (x^2 + z^2)^{-1/2(k+3)} \cos\left(q \operatorname{arc\,tg} \frac{x}{z}\right), \quad \psi = (x^2 + z^2)^{1/2(k+1)} \tag{1.5}$$

$$C = \frac{2^{1+k}\Gamma[1 + 1/2(1+k+q)]\,\Gamma[1 + 1/2(1+k-q)]}{\pi\Gamma(2+k)}$$

$$q = \left[(1+k)\left(1 - \frac{k\nu}{1-\nu}\right)\right]^{1/2}, \quad A = \frac{1(1-\nu)C}{2Kk}$$

Consequently, in the case of a distributed load $p(x)$ the stresses and the displacements are expressed by the integrals

$$\sigma_x = \int_{-\infty}^{\infty} p(\xi) R_1(x, z, \xi)\, d\xi, \quad u_x = \int_{-\infty}^{\infty} p(\xi) K_1(x, z, \xi)\, d\xi$$

$$\sigma_z = \int_{-\infty}^{\infty} p(\xi) R_2(x, z, \xi)\, d\xi, \quad u_z = \int_{-\infty}^{\infty} p(\xi) K_2(x, z, \xi)\, d\xi$$

$$\sigma_{xz} = \int_{-\infty}^{\infty} p(\xi) R_3(x, z, \xi)\, d\xi \tag{1.6}$$

$$R_1(x, z) = Cz^k x^2 \varphi, \quad R_2(x, z) = Cz^{k+2}\varphi, \quad R_3(x, z) = Cz^{k+1} x\varphi$$

$$K_1(x, z) = \frac{-A}{\psi}\left[x \cos\left(q \operatorname{arc\,tg} \frac{x}{z}\right) - \frac{qz}{1+k} \sin\left(q \operatorname{arc\,tg} \frac{x}{z}\right)\right]$$

$$K_2(x, z) = -\frac{A}{\psi}\left[z \cos\left(q \operatorname{arc\,tg} \frac{x}{z}\right) + \frac{qx}{1+k} \sin\left(q \operatorname{arc\,tg} \frac{x}{z}\right)\right]$$

$$R_i(x, z, \xi) = R_i(x - \xi, z) \quad (i = 1, 2, 3), \qquad K_j(x, z, \xi) = K_j(x - \xi, z) \quad (j = 1, 2)$$

φ and ψ according to (1.5)

Let $p^*(r)$ be a locally integrable finite function.
Then

$$P^*(\xi) = \int_{-\infty}^{\xi} p(x)\,dx = o(\xi^{-1}) \quad \text{for } \xi \to \infty \tag{1.7}$$

By virtue of this, we obtain for the stresses and the displacements of the plane problem

$$\sigma_x = -\int_{-\infty}^{\infty} P^*(\xi)\frac{\partial R_1(x,z,\xi)}{\partial \xi}\,d\xi, \qquad u_x = -\int_{-\infty}^{\infty} P^*(\xi)\frac{\partial K_1(x,z,\xi)}{\partial \xi}\,d\xi$$

$$\sigma_z = -\int_{-\infty}^{\infty} P^*(\xi)\frac{\partial R_2(x,z,\xi)}{\partial \xi}\,d\xi, \qquad u_z = -\int_{-\infty}^{\infty} P^*(\xi)\frac{\partial K_2(x,z,\xi)}{\partial \xi}\,d\xi$$

$$\tau_{xz} = -\int_{-\infty}^{\infty} P^*(\xi)\frac{\partial R_3(x,z,\xi)}{\partial \xi}\,d\xi \tag{1.8}$$

We represent $R_i(x,z)$ $(i = 1,2,3)$ and $K_j(x,z)$ $(j = 1,2)$ by the Fourier integrals

$$R_i(x,z) = \int_{-\infty}^{\infty} R_{i*}(z,s)\,e^{isx}ds, \qquad K_j(x,z) = \int_{-\infty}^{\infty} K_{j*}(z,s)\,e^{isx}ds \tag{1.9}$$

Substituting (1.9) into the equalities (1.8), interchanging the order of integration (this is legitimate by virtue of the condition (1.7)), we obtain the stress components as expressed by integrals of the form

$$\int_{-\infty}^{\infty} isR_{i*}(z,s)\,e^{isx}ds \int_{-\infty}^{\infty} P^*(\xi)\,e^{-is\xi}d\xi \qquad (i = 1,2,3) \tag{1.10}$$

and the displacement components by the integrals

$$\int_{-\infty} isK_{j*}(z,s)\,e^{isx}ds \int_{-\infty}^{\infty} P^*(\xi)\,e^{-is\xi}d\xi \qquad (j = 1,2) \tag{1.11}$$

Since

$$P^*(\xi) = \frac{P}{2\pi^2\xi} \quad \text{for} \quad p(x) = -\frac{P}{2\pi^2x^2}$$

the inner integral in (1.10) and (1.11) is equal to $-i(2\pi)^{-1}P \operatorname{sgn} s$.

Then, the solution of the plane problem is

$$\sigma_x = \frac{P}{2\pi}\int_{-\infty}^{\infty} s \operatorname{sgn} s R_{1*}(z,s)\,e^{isx}ds, \qquad \sigma_z = \frac{P}{2\pi}\int_{-\infty}^{\infty} s \operatorname{sgn} s R_{2*}(z,s)\,e^{isx}ds$$

$$\tau_{xz} = \frac{P}{2\pi}\int_{-\infty}^{\infty} s \operatorname{sgn} s R_{3*}(z,s)\,e^{isx}ds \tag{1.12}$$

$$u_x = \frac{P}{2\pi}\int_{-\infty}^{\infty} s \operatorname{sgn} s K_{1*}(z,s)\,e^{isx}ds, \qquad u_z = \frac{P}{2\pi}\int_{-\infty}^{\infty} s \operatorname{sgn} s K_{2*}(z,s)\,e^{isx}ds$$

The obtained results allow us to compute, with the aid of A. Ia. Aleksandrov's transform (1.2), the stresses and the displacements in the axially symmetric case. Hence

$$\sigma_z^* = PI_{20}^{(1)}(r,z), \qquad \sigma_r^* = P\left[I_{10}^{(1)}(r,z) - \frac{1-\nu}{r} I_{11}^{(0)}(r,z) + \frac{\nu}{r} I_{21}^{(0)}(r,z)\right]$$

$$\sigma_\theta^* = \nu P\left[I_{10}^{(1)}(r,z) + I_{20}^{(1)}(r,z) + \frac{1-\nu}{\nu r} I_{11}^{(0)}(r,z) - \frac{1}{r} I_{21}^{(0)}(r,z)\right]$$

$$\tau_{rz}^* = PiI_{31}^{(1)}(r,z) \qquad (1.13)$$

Here we have introduced the following notation:

$$I_{\varkappa n}^{(0)}(r,z) = \int_0^\infty R_{\varkappa *}(z,s) J_n(rs)\,ds, \qquad I_{\varkappa n}^{(1)}(r,z) = \int_0^\infty sR_{\varkappa *}(z,s) J_n(rs)\,ds \qquad (1.14)$$

$$u^* = Pi\int_0^\infty sK_{1*}(z,s)J_1(rs)\,ds, \qquad w^* = P\int_0^\infty sK_{2*}(z,s)J_0(rs)\,ds \qquad (1.15)$$

Inverting (1.9) and inserting the result in (1.13) and (1.15), we obtain the expressions for the stresses and the displacements. Beforehand, in order to avoid a cumbersome expression, we introduce the specially designed notation

$$\gamma_\pm^{(1)} = \frac{1}{\Gamma\left[{}^1/_2(k \pm q + 1)\right]}, \qquad \gamma_\pm^{(3)} = \frac{1}{\Gamma\left[{}^1/_2(k \pm q + 3)\right]} \qquad (1.16)$$

$$B_{\varkappa\lambda\,\mu n}^{\pm}(z,r) = \int_0^\infty s^{1/2(k+\varkappa)} W_{\pm(1/2\,q+\lambda),\,-(1/2\,q+\mu)}(2zs)\,J_n(rs)\,ds$$

Here, the signs in the right and left-hand sides correspond to each other. With this notation, the expressions have the form

for the stresses

$$\sigma_z^* = -CPz^{1/2(k+1)}2^{-1/2(k+5)}\left[\gamma_+^{(3)}B_{3010}^+(r,z) + \gamma_-^{(3)}B_{3010}^-(r,z)\right] \qquad (1.17)$$

$$\tau_{rz}^* = CPz^{1/2(k-1)}2^{-1/2(k+5)}\{[{}^1/_2(1+k+q)\gamma_+^{(3)}B_{1011}^+(z,r) +$$
$$+ {}^1/_2(1+k-q)\gamma_-^{(3)}B_{1011}^-(z,r)] - z[\gamma_+^{(3)}B_{3011}^+(z,r) + \gamma_-^{(3)}B_{3011}^-(z,r)] -$$
$$-[{}^1/_4(k^2-q^2+3)+{}^1/_2q+k]\gamma_+^{(3)}B_{1,-1,11}^+(z,r) - [{}^1/_4(k^2-q^2+3) - {}^1/_2q + k]\times$$
$$\times\gamma_-^{(3)}B_{1111}^-(z,r)\} \qquad (1.18)$$

for the displacements

$$w^*(r,z) = \frac{(1-\nu)CP}{2^{1/2(k+5)}Kk(1+k)z^{1/2(k+1)}}\{z[(1+k-q)\gamma_+^{(1)}B_{1000}^+(r,z) +$$
$$+(1+k+q)\gamma_-^{(1)}B_{1000}^-(r,z)] + {}^1/_2(k+q-1)^q\gamma_+^{(1)}B_{-1000}^+(r,z) -$$
$$-{}^1/_2(k-q-1)^q\gamma_-^{(1)}B_{-1000}^-(r,z) - [{}^1/_4(k^2-q^2-1) + {}^1/_2q]\,q\gamma_+^{(1)}B_{-1,-1,00}^+(r,z) +$$
$$+[{}^1/_4(k^2-q^2-1) - {}^1/_2q]\,q\gamma_-^{(1)}B_{-1100}^-(r,z)\} \qquad (1.19)$$

$$u^*(r,z) = -\frac{(1-\nu)CP}{2^{1/2(k+5)}Kkz^{1/2(k+1)}}\{{}^1/_2(k+q-1)\gamma_+^{(1)}B_{-1001}^+(r,z) +$$
$$+{}^1/_2(k-q-1)\gamma_-^{(1)}B_{-1001}^-(r,z) - z(k+1)^{-1}[(k-q+1)\gamma_+^{(1)}B_{1001}^+(r,z) +$$
$$+(k+q+1)\gamma_-^{(1)}B_{1001}^-] - [{}^1/_4(k^2-q^2-1) + {}^1/_2q]\gamma_+^{(1)}B_{-1,-101}^+(r,z) -$$
$$-[{}^1/_4(k^2-q^2-1) - {}^1/_2q]\gamma_-^{(1)}B_{-1101}^-(r,z) \qquad (1.20)$$

Expressing the Bessel functions $J_0(rs)$ and $J_1(rs)$ by the Whittaker functions $M_{0,0}(2irs)$ and $M_{0,1}(2irs)$ respectively, we arrive in the formulas (1.16) - (1.19) to integrals of products of Whittaker functions, which can be computed in the sense of the principal part [10]. As a result we obtain

$$\sigma_z^*(r,z) = -2^{-(k+5)}CPz^{-2}(\gamma_+^{(3)}\Sigma_{1,2,4,7}^- + \gamma_-^{(3)}\Sigma_{1,2,4,7}^+) \qquad (1.21)$$

$$\Sigma^{\pm}_{1,\varkappa,\lambda,\mu}=\sum_{n=0}^{\infty}a_n\,{}_3F_2\left(\frac{1}{2},\ n+\varkappa,\ k+n+\lambda;\ 1,\ \frac{1}{2}(\mu+k\pm q)+n;\ -\frac{ir}{z}\right)$$

Here and in the following we have the notation

$$a_n\,{}_3F_2\left(\alpha_1,\alpha_2,\alpha_3;\ \beta_1,\beta_2;\ -\frac{ir}{z}\right)=\frac{\left(1+\frac{ir}{z}\right)^n}{2^n n!}\,\frac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\beta_2)}\times$$
$$\times\,{}_3F_2\left(\alpha_1,\alpha_2,\alpha_3;\ \beta_1,\beta_2;\ -\frac{ir}{z}\right)$$

$$\tau_{rz}{}^*=2^{-(k+6)}CPrz^{-3}\{{}^1/_2(1+k+q)\gamma_+^{(3)}\Sigma^-_{3,2,4,7}+{}^1/_2(1+k-q)\gamma_-^{(3)}\Sigma^+_{3,2,4,7}-$$
$$-{}^1/_2\gamma_+^{(3)}\Sigma^-_{3,3,5,9}-{}^1/_2\gamma_-^{(3)}\Sigma^+_{3,3,5,9}-[{}^1/_4(k^2-q^2+3)+{}^1/_2q+k]\,\gamma_+^{(3)}\Sigma^-_{3,2,4,9}-$$
$$-[{}^1/_4(k^2-q^2+3)+{}^1/_2q+k]\,\gamma_-^{(3)}\Sigma^+_{3,2,4,9}\} \qquad (1.22)$$

$$\Sigma^{\pm}_{3,\varkappa,\lambda,\mu}=\sum_{n=0}^{\infty}a_n\,{}_3F_2\left(\frac{3}{2},n+\varkappa,\ k+n+\lambda;\ 3,\ \frac{1}{2}(\mu+k\pm q)+n;\ -\frac{iz}{z}\right)$$

where $r<z$. For the same domain of values of r and z the displacements are:

$$w^*(r,z)=(1-\nu)\,CP\,[2^{k+3}Kk(1+k)\,z^{k+1}]^{-1}\{{}^1/_2(1+k-q)\gamma_+^{(1)}\Sigma^-_{1,2,2,5}+$$
$$+{}^1/_2(1+k+q)\gamma_-^{(1)}\Sigma^+_{1,2,2,5}+{}^1/_2(k+q-1)\,q\gamma_+^{(1)}\Sigma^-_{1,1,1,3}-$$
$$-{}^1/_2(k-q-1)\,q\gamma_-^{(1)}\Sigma^+_{1,1,1,3}-[{}^1/_4(k^2-q^2-1)+{}^1/_2q]\,q\gamma_+^{(1)}\Sigma^-_{1,1,1,5}+$$
$$+[{}^1/_4(k^2-q^2-1)-{}^1/_2q]\,q\gamma_-^{(1)}\Sigma^+_{1,1,1,5}\} \qquad (1.23)$$

$$u^*(r,z)=-(1-\nu)\,CPr\,[2^{k+5}Kkz^{k+2}]^{-1}\{{}^1/_2(k+q-1)\gamma_+^{(1)}\Sigma^-_{3,2,2,5}+$$
$$+{}^1/_2(k-q-1)\gamma_-^{(1)}\Sigma^+_{3,2,2,5}-{}^1/_2(1+k-q)(1+k)^{-1}\gamma_+^{(1)}\Sigma^-_{3,3,3,7}-$$
$$-{}^1/_2(1+k+q)(1+k)^{-1}\gamma_-^{(1)}\Sigma^+_{3,3,3,7}-[{}^1/_4(k^2-q^2-1)+{}^1/_2q]\,\gamma_+^{(1)}\Sigma^-_{3,2,2,7}-$$
$$-[{}^1/_4(k^2-q^2-1)-{}^1/_2q]\,\gamma_-^{(1)}\Sigma^+_{3,2,2,7}\} \qquad (1.24)$$

However, if in the equalities (1.16) - (1.19) we express the Whittaker function $W_{\lambda,\mu}(2zs)$ in terms of the Whittaker functions $M_{\lambda,\pm\mu}(2zs)$, and the Bessel functions $J_0(rs)$ and $J_1(rs)$ in terms of the Whittaker functions $W_{0,0}(\pm 2irs)$ and $W_{0,1}(\pm 2irs)$ respectively, then, making use of the same formula (7.625 (1) [10]), we obtain for the domain $z<r$

$$\sigma_z{}^*=-2^{-(k+5)}CP\pi^{-1/2}\{i\Gamma(k+2)\gamma_+^{(3)}\gamma_-^{(3)}[S^+_{-(1+k),2,5/2}-S'^+_{-(1+k),2,5/2}]+$$
$$+\Gamma(-k-2)\gamma_+^{(3)}\Gamma^{-1}[{}^1/_2(-1-k-q)]\,z^{k+2}r^{-(k+4)}[e^{1/2(k+3)\pi i}S^+_{(k+3),(k+4),(k+9/2)}+$$
$$+l^{-1/2(k+3)\pi i}S'^+_{(k+3),(k+4),(k+9/2)}]+i\Gamma(k+2)\gamma_+^{(3)}\gamma_-^{(3)}[S^-_{-(1+k),2,5/2}-S'^-_{-(1+k),2,5/2}]+$$
$$+\Gamma(-k-2)\gamma_-^{(3)}\Gamma^{-1}[{}^1/_2(-1-k+q)]\,z^{k+2}r^{-(k+4)}\times$$
$$\times[e^{1/2(k+3)\pi i}S^-_{(k+3),(k+4),(k+9/2)}+e^{-1/2(k+3)\pi i}S'^-_{(k+3),(k+4),(k+9/2)}]\} \qquad (1.25)$$

$$S^{\pm}_{\varkappa,\lambda,\mu}=\sum_{n=0}^{\infty}b_n\,{}_3F_2\left({}^1/_2(\pm q+\varkappa),\ \ n+\lambda,\ n+\lambda;\ \varkappa,\ n+\mu;\ \frac{z}{ir}\right)$$

$$S'^{\pm}_{\varkappa,\lambda,\mu}=\sum_{n=0}^{\infty}C_n\,{}_3F_2\left({}^1/_2(\pm q+\varkappa),\ n+\lambda,\ n+\lambda;\ \varkappa,n+\mu;\ -\frac{z}{ir}\right).$$

Here and in the following

$$b_n\, {}_3F_2\left(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; \frac{z}{ir}\right) = \frac{\left(1-\frac{z}{ir}\right)^n}{2^n n!} \frac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\beta_2)}\; {}_3F_2\left(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; \frac{z}{ir}\right)$$

$$c_n\, {}_3F_2\left(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; -\frac{z}{ir}\right) = \frac{\left(1+\frac{z}{ir}\right)^n \Gamma(\alpha_2)\Gamma(\alpha_3)}{2^n n!\, \Gamma(\beta_2)} \times$$
$$\times\, {}_3F_2\left(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; -\frac{z}{ir}\right)$$

$$\tau_{rz}{}^* = 2^{-(k+3)} CP\pi^{-1/2} \{{}^1/_2 (1+k+q)\,\Gamma(k+2)\,\gamma_+{}^{(3)}\gamma_-{}^{(3)} (rz)^{-1}\, i\, [N'^{+}_{-(1+k),\,2,\,3/2} -$$
$$- N^{-}_{-(1+k),\,2,\,3/2}] + {}^1/_2 (1+k+q)\,\Gamma(-k-2)\,\gamma_+{}^{(3)}\,\Gamma^{-1}[{}^1/_2(-1-k-q)] \times$$
$$\times\, r^{-(k+3)} z^{k+1} [e^{-1/2(k+1)\pi i} N^{1+}_{(3+k),\,(k+4),\,(k+7/2)} + e^{1/2(k+1)\pi i} N^{+}_{(k+3),\,(k+4),\,(k+7/2)}] +$$
$$+ {}^1/_2 (1+k-q)\,\Gamma(k+2)\gamma_+{}^{(3)}\gamma_-{}^{(3)}(rz)^{-1}\, i\, [N'^{-}_{-(1+k),\,2,\,3/2} - N^{-}_{-(1+k),\,2,\,3/2}] +$$
$$+ {}^1/_2 (1+k-q)\,\gamma_-{}^{(3)}\,\Gamma(-k-2)\,\Gamma^{-1}[{}^1/_2(-1-k+q)]\, z^{k+1} r^{-(k+3)} \times$$
$$\times\, [e^{-1/2(k+1)\pi i} N'^{-}_{(k+3),\,(k+4)\,(k+7/2)} + c^{1/2(k+1)\pi i} N^{-}_{(k+3),\,(k+4),\,(k+7/2)}] - {}^1/_2\,\Gamma(k+2) \times$$
$$\times\, \gamma_+{}^{(3)}\gamma_-{}^{(3)} r^{-2}\, i\, [N'^{+}_{-(1+k),\,3,\,5/2} - N^{+}_{-(1+k),\,3,\,5/2}] - {}^1/_2\Gamma(-k-2)\Gamma^{-1}[{}^1/_2(-1-k-q)] \times$$
$$\times\, \gamma_+{}^{(3)} z^{k+2} r^{-(k+4)} [e^{-1/2(k+2)\pi i} N'^{+}_{(k+3),\,(k+5),\,(k+9/2)} + e^{1/2(k+2)\pi i} N^{+}_{(k+3),\,(k+5),\,(k+9/2)}] -$$
$$- {}^1/_2\,\Gamma(k+2)\,\gamma_+{}^{(3)}\gamma_-{}^{(3)}\, i\, [N'^{-}_{-(1+k),\,3,\,5/2} - N^{-}_{-(1+k),\,3,\,5/2}] - {}^1/_2\,\Gamma(-k-2) \times$$
$$\times\, \Gamma^{-1}[{}^1/_2(-1-k+q)]\,\gamma_-{}^{(3)} z^{k+2} r^{-(k+4)} [e^{-1/2(k+2)\pi i} N'^{-}_{(k+3),\,(k+5),\,(k+9/2)} + e^{1/2(k+2)\pi i} \times$$
$$\times\, N^{-}_{(k+3),\,(k+5),\,(k+9/2)}] - [{}^1/_4(k^2-q^2+3) + {}^1/_2 q + k]\,\Gamma(k+2)\,\Gamma^{-1}[{}^1/_2(5+$$
$$+ k + q)]\,\gamma_+{}^{(3)} (rz)^{-1}\, i\, [M'^{+}_{-(3+k),\,2,\,3/2} - M^{+}_{-(3+k),\,2,\,3/2}] - [{}^1/_4(k^2-q^2+3+$$
$$+ {}^1/_2 q + k)]\,\Gamma(-k-2)\,\Gamma^{-1}[{}^1/_2(1-k-q)\,\gamma_+{}^{(3)} z^{k+1} r^{-(3+k)} [e^{-1/2(k+1)\pi i} \times$$
$$\times\, M'^{+}_{(k+1),\,(k+4),\,(k+7/2)} + e^{1/2(k+1)\pi i} M^{+}_{(1+k),\,(k+4),\,(k+7/2)}] - [{}^1/_4(k^2-q^2+3) -$$
$$- {}^1/_2 q + k]\,\Gamma(k+2)\,\Gamma^{-1}[{}^1/_2(5+k+q]\,\gamma_+{}^{(3)} (rz)^{-1}\, i \times [M'^{-}_{-(k+3),\,2,\,3/2} -$$
$$- M^{-}_{-(k+3),\,2,\,3/2}] - [{}^1/_4(k^2-q^2+3) - {}^1/_2 q + k]\,\Gamma(-k-2)\,\Gamma^{-1}[{}^1/_2(1-$$
$$- k + q)]\,\gamma_-{}^{(3)} z^{k+1} r^{-(k+3)} [e^{-1/2(k+1)\pi i} M'^{-}_{(1+k),\,(k+4),\,(k+7/2)} + e^{1/2(k+1)\pi i} \times$$
$$\times\, M^{-}_{(k+1),\,(k+4),\,(k+7/2)}] \tag{1.26}$$

$$N^{\pm}_{\varkappa,\,\lambda,\,\mu} = \sum_{n=0}^{\infty} b_n\, {}_3F_2\left({}^1/_2(\pm q + \varkappa),\; n+\lambda,\; n+(\lambda-2);\; \varkappa,\; n+\mu; \frac{z}{ir}\right)$$

$$N'^{\pm}_{\varkappa,\,\lambda,\,\mu} = \sum_{n=0}^{\infty} C_n\, {}_3F_2\left({}^1/_2(\pm q + \varkappa),\; n+\lambda,\; n+(\lambda-2);\; \varkappa,\; n+\mu;\; -\frac{z}{ir}\right)$$

$$M^{\pm}_{\varkappa,\,\lambda,\,\mu} = \sum_{n=0}^{\infty} b_n\, {}_3F_2\left({}^1/_2(\pm q + \varkappa),\; n+\lambda,\; n+(\lambda-2);\; \varkappa+2,\; n+\mu;\; \frac{z}{ir}\right)$$

$$M'^{\pm}_{\varkappa,\,\lambda,\,\mu} = \sum_{n=0}^{\infty} C_n\, {}_3F_2\left({}^1/_2(\pm q + \varkappa),\; n+\lambda,\; n+(\lambda-2);\; \varkappa+2,\; n+\mu;\; \frac{z}{ir}\right)$$

The displacements in the domain $z < r$ are

$$w^*(r, z) = (1-\nu)\, CP\, [2^{k+3} K k (1+k) \sqrt{\pi}]^{-1} \{{}^1/_2 (1+k-q)\,\Gamma(k)\,\gamma_+{}^{(1)}\gamma_-{}^{(1)} \times$$

$$\times z^{1-k} r^{-2} i \,[S'^{+}_{(1-k),\,2,\,5/2} + S^{+}_{(1-k),\,2,\,5/2}] + (1 + k - q)\,\Gamma(-k)\,\gamma_{+}^{(1)} \times$$
$$\times \Gamma^{-1}[{}^{1}/_{2}(1 - k - q)]\, z r^{-2} i\,[S'^{+}_{(1+k),\,(k+2),\,(k+5/2)} + S^{+}_{(1+k),\,(k+2),\,(k+5/2)}] + {}^{1}/_{2}(1 +$$
$$+ k + q)\,\Gamma(k)\,\gamma_{+}^{(1)}\gamma_{-}^{(1)} z^{1-k} r^{-2} i\,[S'^{-}_{(1-k),\,2,\,5/2} + S^{-}_{(1-k),\,2,\,5/2}] + (1 + k + q)\,\gamma_{-}^{(1)} \times$$
$$\times \Gamma(-k)\,\Gamma^{-1}[{}^{1}/_{2}(1 - k + q)]\, z r^{-2} i\,[S'^{-}_{(1+k),\,(k+2),\,(k+5/2)} + S^{-}_{(1+k),\,(k+2),\,(k+5/2)}] +$$
$$+ {}^{1}/_{2}(k + q - 1)\, q\,\Gamma(k)\,\gamma_{+}^{(1)}\gamma_{-}^{(1)} z^{-k} r^{-1}\,[S^{+}_{(1-k),\,1,\,5/2} - S'^{+}_{(1-k),\,1,\,5/2}] +$$
$$+ {}^{1}/_{2}(k + q - 1)\, q\,\Gamma(-k)\,\gamma_{+}^{(1)}\Gamma^{-1}[{}^{1}/_{2}(1 - k - q)]\, r^{-(1+k)}\,[e^{-k\pi/2i} \times$$
$$\times S'^{+}_{(k+1),\,(k+1),\,(k+3/2)} + e^{1/2\,k\pi i} S^{+}_{(k+1),\,(k+1),\,(k+3/2)})] - {}^{1}/_{2}(k - q - 1)\, q\,\Gamma(k)\,\gamma_{-}^{(1)}\gamma_{+}^{(1)} \times$$
$$\times z^{-k} r^{-1}\,[S^{-}_{(1-k),\,1,\,5/2} - S'^{-}_{(1-k),\,1,\,5/2}] - {}^{1}/_{2}(k - q - 1)\, q\,\Gamma(-k)\,\gamma_{-}^{(1)}\Gamma^{-1}[{}^{1}/_{2}(1 -$$
$$- k + q)]\, r^{-(1+k)}\,[e^{-1/2\,k\pi i}\, S'^{-}_{(k+1),\,(k+1),\,(k+3/2)} + e^{1/2\,k\pi i} S^{-}_{(k+1),\,(k+1)\,(k+3/2)}] -$$
$$- [{}^{1}/_{4}(k^{2} - q^{2} - 1) + {}^{1}/_{2}q]\, q\,\Gamma(k)\,\gamma_{+}^{(1)}\gamma_{-}^{(3)} z^{-k} r^{-1}\,[L^{+}_{-(1+k),\,1,\,5/2} - L'^{-}_{-(1+k),\,1,\,5/2}] -$$
$$- [{}^{1}/_{4}(k^{2} - q^{2} - 1) + {}^{1}/_{2}q]\, q\,\Gamma(-k)\,\gamma_{+}^{(1)}\Gamma^{-1}[{}^{1}/_{2}(3 - k - q)]\, r^{-(k+1)} \times$$
$$\times [e^{-1/2\,k\pi i}\, L'^{+}_{(k-1),\,(k+1),\,(k+5/2)} + e^{1/2\,k\pi i}\, L^{+}_{(k-1),\,(k+1),\,(k+3/2)}] +$$
$$+ [{}^{1}/_{4}(k^{2} - q^{2} - 1) - {}^{1}/_{2}q]\, q\,\Gamma(k)\,\gamma_{-}^{(1)}\gamma_{+}^{(3)} z^{-k}\,[L^{-}_{-(k+1),\,1,\,3/2} - L'^{-}_{(k+1),\,1,\,5/2}] +$$
$$+ [{}^{1}/_{4}(k^{2} - q^{2} - 1) - {}^{1}/_{2}q]\, q\,\Gamma(-k)\,\gamma_{-}^{(1)}\Gamma^{-1}[{}^{1}/_{2}(3 - k + q)\, r^{-(k+1)}\,[e^{-1/2\,k\pi i} \times$$
$$\times L'^{-}_{(k-1),\,(k+1,\,(k+3/2)} + e^{1/2\,k\pi i} L^{-}_{(k-1),\,(k+1),\,(k+3/2)}]\} \qquad (1.27)$$

$$L^{\pm}_{\varkappa,\,\lambda,\,\mu} = \sum_{n=0}^{\infty} b_{n}\,{}_{3}F_{2}\left({}^{1}/_{2}(\pm q + \varkappa),\ n + \lambda,\ n + \lambda;\ \varkappa + 2;\ n + \mu;\ \frac{z}{ir}\right)$$

$$L^{\pm}_{\varkappa,\,\lambda,\,\mu} = \sum_{n=0}^{\infty} C_{n}\,{}_{3}F_{2}\left({}^{1}/_{2}(\pm q + \varkappa,\ n + \lambda,\ n + \lambda;\ \varkappa + 2,\ n + \mu;\ -\frac{z}{ir}\right)$$

$$u^{*}(r, z) = -(1 - \nu) CP\,(2^{k+3} K k \sqrt{\pi})^{-1}\{{}^{1}/_{2}(k + q - 1)\,\Gamma(k)\,\gamma_{+}^{(1)}\gamma_{-}^{(1)} z^{-k} r^{-1} i \times$$
$$\times [N'^{+}_{(1-k),\,2,\,5/2} - N^{+}_{(1-k),\,2,\,5/2}] + {}^{1}/_{2}(k + q - 1)\,\Gamma(-k)\,\gamma_{+}^{(1)}\Gamma^{-1}[{}^{1}/_{2}(1 - k - q)] \times$$
$$\times r^{-(k+1)}\,[e^{1/2(1-k)\pi i}\, N'^{+}_{(k+1),\,(k+2),\,(k+3/2)} + e^{-1/2(1-k)\pi i} N^{+}_{(k+1),\,(k+2),\,(k+3/2)}] +$$
$$+ {}^{1}/_{2}(k - q - 1)\,\Gamma(k)\,\gamma_{-}^{(1)}\gamma_{+}^{(1)} z^{-k} r^{-1} i\,[N'^{-}_{(1-k),\,2,\,5/2} - N^{-}_{(1-k),\,2,\,5/2}] +$$
$$+ {}^{1}/_{2}(k - q - 1)\,\Gamma(-k)\,\gamma_{-}^{(1)}\Gamma^{-1}[{}^{1}/_{2}(1 - k + q)]\, r^{-(k+1)}\,[e^{1/2(1-k)\pi i} \times$$
$$\times N'^{-}_{(k+1),\,(k+2),\,(k+3/2)} + e^{-1/2(1-k)\pi i} N^{-}_{(k+1),\,(k+2),\,(k+3/2)}] - {}^{1}/_{2}(1 + k - q)(1 + k)^{-1} \times$$
$$\times \Gamma(k)\,\gamma_{+}^{(1)}\gamma_{-}^{(1)} z^{1-k} r^{-2}\,[N'^{+}_{(1-k),\,3,\,5/2} + N^{+}_{(1-k),\,3,\,5/2}] - {}^{1}/_{2}(1 + k - q)(1 + k)^{-1} \times$$
$$\times \Gamma(-k)\,\gamma_{+}^{(1)}\Gamma^{-1}[{}^{1}/_{2}(1 - k - q)]\, z r^{-(k+2)}\,[e^{-1/2\,k\pi i} N'^{+}_{(1+k),\,(k+3),\,(k+5/2)} +$$
$$+ e^{1/2\,k\pi i} N^{+}_{(k+1),\,(k+3),\,(k+5/2)}] - {}^{1}/_{2}(1 + k + q)(1 + k)^{-1}\,\Gamma(k)\,\gamma_{-}^{(1)}\gamma_{+}^{(1)} z^{1-k} r^{-2} \times$$
$$\times [N'^{-}_{(1-k),\,3,\,5/2} + N^{-}_{(1-k),\,3,\,5/2}] - {}^{1}/_{2}(1 + k + q)(1 + k)^{-1}\,\Gamma(-k)\,\gamma_{-}^{(1)} \times$$
$$\times \Gamma^{-1}[{}^{1}/_{2}(1 - k + q)]\, z\, r^{-(k+2)}\,[e^{-1/2\,k\pi i}\, N'_{(1+k),\,(k+3),\,(k+5/2)} + e^{1/2\,k\pi i} N^{-}_{(k+1),\,(k+3),\,(k+5/2)}] -$$
$$- [{}^{1}/_{4}(k^{2} - q^{2} - 1) + {}^{1}/_{2}q]\,\Gamma(k)\,\gamma_{+}^{(1)}\gamma_{-}^{(3)} z^{-k} r^{-1} i \times$$
$$\times [E'^{+}_{-(k+1),\,2,\,3/2} - E^{+}_{-(k+1),\,2,\,5/2}] - [{}^{1}/_{4}(k^{2} - q^{2} - 1) + {}^{1}/_{2}q]\,\Gamma(-k)\,\gamma_{+}^{(1)}\Gamma^{-1}[{}^{1}/_{2}(3 -$$
$$- k - q)]\, r^{-(k+1)}\,[e^{1/2(1-k)\pi i}\, E'^{+}_{(k-1),\,(k+2),\,(k+3/2)} + e^{-1/2(1-k)\pi i}\, E^{+}_{(k-1),\,(k+2),\,(k+3/2)}] -$$
$$- [{}^{1}/_{4}(k^{2} - q^{2} - 1) - {}^{1}/_{2}q]\,\Gamma(k)\,\gamma_{-}^{(1)}\gamma_{+}^{(3)} z^{-k} i\,[E'^{-}_{-(k+1),\,2,\,3/2} - E^{-}_{-(k+1),\,2,\,3/2}] - [{}^{1}/_{4}(k^{2} -$$
$$- q^{2} - 1) - {}^{1}/_{2}q]\,\Gamma(-k)\,\gamma_{-}^{(1)}\Gamma^{-1}[{}^{1}/_{2}(3 - k + q)]\, r^{k-1}\,[e^{1/2(1-k)\pi i}\, E'^{-}_{(k-1),\,(k+2),\,(k+3/2)} +$$
$$+ e^{-1/2(1-k)\pi i}\, E^{-}_{(k-1),\,(k+2),\,(k+3/2)}]\} \qquad (1.28)$$

$$E^{\pm}_{\varkappa,\lambda,\mu} = \sum_{n=0}^{\infty} b_n\, {}_3F_2\left(\tfrac{1}{2}(\pm q + \varkappa), n + \lambda, n + (\lambda - 2); \varkappa + 2, n + \mu; \frac{z}{ir}\right)$$

$$E'^{\pm}_{\varkappa,\lambda,\mu} = \sum_{n=0}^{\infty} C_n\, {}_3F_2\left(\tfrac{1}{2}(\pm q + \varkappa), n + \lambda, n + (\lambda - 2); \varkappa + 2, n + \mu; -\frac{z}{ir}\right)$$

Because of their awkwardness, we do not give the formulas for σ_r^* and σ_θ^*. The derivation of these formulas from the equalities (1.12), (1.13) is similar to the one given above.

Thus, the solution of the formulated problem can be represented in the entire half-space in integral form by the formulas (1.16) - (1.19), which are expressed by analytic functions in the form of series of generalized hypergeometric functions (1.12) - (1.24) for the domain $r < z$, and (1.25) - (1.28) for the domain $r > z$. The functions, representing the solution in different domains, are not the analytic continuations of each other.

2. From the integral form of the solution one can obtain an asymptotic expansion for the obtained solution. Thus, making use of the asymptotic expansion of the Whittaker function, we obtain an asymptotic expansion for σ_z^* for large z and a fixed r

$$\sigma_z^* \sim -CPz^{-2}\left[\frac{3+k+q}{2^{1/2(7+k-q)}} F\left(\frac{5+k+q}{4}, \frac{7+k+q}{4}; 1; -\frac{r^2}{z^2}\right) + \right.$$
$$\left. + \frac{3+k-q}{2^{1/2(7-k+q)}} F\left(\frac{5+k-q}{4}, \frac{7+k-q}{4}; 1; -\frac{r^2}{z^2}\right)\right] \qquad (2.1)$$

For small values of z and fixed r we have the following asymptotic expansion:

$$\sigma_z^* \sim -CP\left\{\frac{2^{-(k+2)}\Gamma(k+2)}{\gamma_+^{(3)}\gamma_-^{(3)}}\frac{r}{(z^2+r^2)^{3/2}} + \left[\frac{\Gamma(-k-2)\Gamma(k+4)}{\gamma_+^{(3)}\Gamma[1/2(-1-k-q)]} + \right.\right.$$
$$\left.\left. + \frac{\Gamma(-k-2)\Gamma(k+4)}{\gamma_-^{(3)}\Gamma[1/2(-1-k+q)]}\right]\frac{z^{k+2}}{2(z^2+r^2)^{1/2(k+4)}} F\left(\frac{k+4}{2}, \frac{-k-3}{2}; 1; \frac{r^2}{z^2+r^2}\right)\right\} \qquad (2.2)$$

From the last relation it follows that $\sigma_z^* \to 0$ as $z \to 0$ and $r \neq 0$. The fact that the obtained solution satisfies the boundary conditions (1.28), can be easily seen also from the exact representation of the solution by the formulas (1.23), (1.24), since

$$\sum_{n=0}^{\infty} c_n\, {}_3F_2\left(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; -\frac{z}{ir}\right)$$

$$\sum_{n=0}^{\infty} b_n\, {}_3F_2\left(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2; \frac{z}{ir}\right) \qquad \text{for } z \to 0$$

tend to the same limit and the generalized hypergeometric functions which occur in the sums have identical parameters. This limit is equal to

$$\frac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\beta_2)}\, {}_2F_1(\alpha_2, \alpha_3; \beta_2; 1/2)$$

From (1.27) we have for $r = 0$ that $\sigma_z^* \sim Az^{-2}$, where A is some constant The same behavior can be obtained for σ_z^* on the z-axis from the exact representation of the solution by formula (1.20), namely

$$\sigma_z^*|_{r=0} = -\frac{3P}{2\pi}Mz^{-2}$$

Here

$$M = \frac{(k+3)(k+2)}{6}\left[\frac{1}{(5+k-q)(3+k-q)}F\left(2, k+4; \frac{7+k-q}{2}; \frac{1}{2}\right) + \right.$$
$$\left. + \frac{1}{(5+k+q)(3+k+q)}F\left(2, k+4; \frac{7+k+q}{2}; \frac{1}{2}\right)\right]$$

We give the values of the constant M, computed for some values of ν and k

ν	k	q	M
	$^1/_8$	1.026980	1.063809
$^1/_3$	$^1/_2$	1.060670	1.215811
	$^7/_8$	1.026980	1.308136
	$^1/_8$	1.037821	1.030632
$^1/_4$	$^1/_2$	1.118034	1.349478
	$^7/_8$	1.153698	1.452385

On the boundary plane $z = 0$ the displacements are obtained in the form

$$w^*(r, 0) = \frac{(1-\nu)CPq\sin(^1/_2\pi q)\,\Gamma(^1/_2 + {}^1/_2k)}{4K(1+k)r^{1+k}\sqrt{\pi}\,\Gamma(1+{}^1/_2k)}$$

$$u^*(r, 0) = \frac{(1-\nu)CP\cos(^1/_2\pi q)\,\Gamma(1+{}^1/_2k)}{2Kkr^{1+k}\sqrt{\pi}\,\Gamma(^1/_2+{}^1/_2k)} \tag{2.3}$$

The first of the formulas (2.3) has been obtained earlier [11]. For $k = 0$ from (1.20) we obtain

$$\sigma_z^*(r, z) = -\frac{P}{2^4\pi z^2}\left[\sum_{n=0}^{\infty} a_{n3}F_2\left(^1/_2, n+2, n+4; 1, n+3; -\frac{ir}{z}\right) + \right.$$
$$\left. + \sum_{n=0}^{\infty} a_{n3}F_2\left(^1/_2, n+2, n+4; 1, n+4; -\frac{ir}{z}\right)\right].$$

Hence, for $r = 0$ we obtain the already known result [12]

$$\sigma_z^*(0, z) = -3P(2\pi z^2)^{-1}$$

The already known results [12] can be obtained also from the formula (2.1) for $k = 0$.

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CLOSED FORM SOLUTIONS OF PROBLEMS ON THE ELASTIC EQUILIBRIUM OF AN INFINITE WEDGE WITH NONSYMMETRIC NOTCH AT THE APEX

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The solution is given of a number of problems on the elastic equilibrium of an infinite wedge with a nonsymmetric notch at the apex by using the method elucidated in [1]. The solution is obtained in the form of the Cauchy-type integrals for various homogeneous conditions on the side faces of the wedge.

The problem of representing 2×2 matrices given on a curve L in the complex plane and belonging to a certain class is posed and solved in closed form in [2, 3] in the form of the product of 2×2 matrices holomorphic to the left and right of L, whose boundary conditions on L commutate.

A simpler solution of the mentioned homogeneous Hilbert problem, more convenient for applications, is given in [1]. It is also shown here that the problem of elastic equilibrium of an infinite wedge with nonsymmetric notch at the apex and stress-free faces reduces to an inhomogeneous Hilbert problem for a two-dimensional piecewise-holomorphic vector, where the matrix factor belongs to the above-mentioned class in three cases.

1. Reduction of the problem of elastic equilibrium of a wedge with a notch to an inhomogeneous Hilbert problem. Let an infinite triangular wedge occupy the domain $0 \leqslant \varphi \leqslant \theta$ in a plane with the polar coordinates r, φ. Values of the stresses $\sigma_\varphi, \tau_{r\varphi}$ are given on the face $\varphi = 0$, but on the face $\varphi = \theta$ we consider homogeneous conditions of one of the following kinds :